Problem Sheet 1

1. Let $\Omega = \{1, 2, 3\}$. Let

$$\mathcal{F} = \{\emptyset, \{1\}, \{2, 3\}, \{1, 2, 3\}\},$$
$$\mathcal{F}' = \{\emptyset, \{2\}, \{1, 3\}, \{1, 2, 3\}\}.$$

You may assume that both $\mathcal{F}$ and $\mathcal{F}'$ are σ-fields.

(a) Show that $\mathcal{F} \cup \mathcal{F}'$ is not a σ-field.

(b) Let $X : \Omega \to \mathbb{R}$ be defined by

$$X(n) = \begin{cases} 1 & \text{if } n = 1 \\ 2 & \text{if } n = 2 \\ 1 & \text{if } n = 3 \end{cases}$$

Is X measurable with respect to $\mathcal{F}$? Is X measurable with respect to $\mathcal{F}'$?

2. Let Ω be any set. Let I be any set and for each $i \in I$ let $\mathcal{F}_i$ be a σ-field on Ω. Prove that $\cap_{i \in I} \mathcal{F}_i$ is a σ-field on Ω.

3. Let $\Omega = \{0, 1\}^{\mathbb{N}}$, and let us write each $\omega \in \Omega$ as a sequence: $\omega = \omega_1 \omega_2 \omega_3 \dots$ where $\omega_i \in \{0, 1\}$. Let $\mathcal{F} = \sigma(\{\omega\,;\, \omega_n = C\}\,;\, n \in \mathbb{N}, C \in \{0, 1\})$.

For each $n \in \mathbb{N}$ let $X_n : \Omega \to \mathbb{R}$ be given by $X_n(\omega) = \omega_n$ and define

$$S_n = \sum_{i=1}^{n} X_i.$$

You can think of this setting as follows. We toss a coin infinitely many times. We set $X_i = 0$ if the i^{th} flip is a head and $X_i = 1$ is the i^{th} flip is a tail. Then Ω is the set of all possible outcomes and $\mathcal{F}$ is smallest σ-field for which, for each $i \in \mathbb{N}$, the event that the i^{th} flip is a head (resp. tail) is measurable. Then, S_n is the total number of tails in the first n tosses.

Let $p \in [0, 1]$.

(a) Prove that $\{\frac{S_n}{n} \le p\}$ is $\mathcal{F}$ measurable. (You may assume that linear combinations of measurable functions are measurable.)

(b) Let $m \in \mathbb{N}$. Prove that $\{\sup_{n \ge m} \frac{S_n}{n} \le p\}$ and $\{\sup_{n \ge m} \frac{S_n}{n} < p\}$ are $\mathcal{F}$ measurable.

(c) Prove that $\{\limsup_{n \to \infty} \frac{S_n}{n} = p\}$ is $\mathcal{F}$ measurable.

Suppose additionally that $\mathbb{P} : \mathcal{F} \to [0, 1]$ is a probability measure, under which the X_i are independent and identically distributed with $\mathbb{P}[X_1 = 0] = \mathbb{P}[X_1 = 1] = \frac{1}{2}$.

(d) Calculate $\mathbb{E}[S_2 | \sigma(X_1)]$ and $\mathbb{E}[S_2^2 \,|\, \sigma(X_1)]$.

(e) Let $n \in \mathbb{N}$. Calculate $\mathbb{E}[X_1 | \sigma(S_n)]$.

4. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $X \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$.

(a) Prove that $\mathbb{E}[\mathbb{E}[X \,|\, \mathcal{G}]] = \mathbb{E}[X]$ almost surely.

(b) Let $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Prove that there exists $c \in \mathbb{R}$ such that $\mathbb{E}[X | \mathcal{F}_0] = c$ almost surely. Hence, show that $c = \mathbb{E}[X]$.

5. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $X, Y \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$. Suppose that $\mathbb{E}[X \,|\, \mathcal{G}] = Y$ and $\mathbb{E}[X^2] = \mathbb{E}[Y^2]$. Prove that $X = Y$ almost surely.

Problem Sheet 2

1. Let $(X_n)_{n\in\mathbb{N}}$ be an iid sequence of random variables such that $\mathbb{P}[X_1 = -1] = \mathbb{P}[X_1 = 1] = \frac{1}{2}$. Let
$$S_n = \sum_{i=1}^{n} X_i.$$
Let $\mathcal{F}_n = \sigma(X_i\,;\, i \le n)$.
   (a) Show that $\mathcal{F}_n$ is a filtration and that S_n is a $\mathcal{F}_n$ martingale.
   (b) State, with proof, which of the following processes are $\mathcal{F}_n$ martingales:
$$\text{(i) } S_n^2 \qquad \text{(ii) } S_n^2 - n \qquad \text{(iii) } \frac{S_n}{n}$$
   Which of the above are submartingales?

2. Let $X_0, X_1, \dots$ be a sequence of $\mathcal{L}^1$ random variables. Let $\mathcal{F}_n$ be a filtration and suppose that $\mathbb{E}[X_{n+1}|\mathcal{F}_n] = aX_n + bX_{n-1}$ for all $n \in \mathbb{N}$, where $a, b > 0$ and $a + b = 1$.
Find a value of $\alpha \in \mathbb{R}$ for which $S_n = \alpha X_n + X_{n-1}$ is an $\mathcal{F}_n$ martingale.

3. At time 0, an urn contains 1 black ball and 1 white ball. At each time $n = 1, 2, 3, \dots,$, a ball is chosen from the urn and returned to the urn. At the same time, a new ball of the same colour as the chosen ball is added to the urn. Just after time n, there are $n + 2$ balls in the urn, of which $B_n + 1$ are black, where B_n is the number of black balls added into the urn at or before time n.
Let
$$M_n = \frac{B_n + 1}{n + 2}$$
be the proportion of balls in the urn that are black, at time n. Note that $M_n \in [0, 1]$.
   (a) Show that, relative to a natural filtration that you should specify, M_n is a martingale.
   (b) Calculate the probability that the first k balls drawn are all black and that the next j balls drawn are all white.
   (c) Show that $\mathbb{P}[B_n = k] = \frac{1}{n+1}$ for all $0 \le k \le n$, and deduce that $\lim_{n\to\infty} \mathbb{P}[M_n \le p] = p$ for all $p \in [0, 1]$.
   (d) Let T be the number of balls drawn until the first black ball appears. Show that T is a stopping time and use the Optional Stopping Theorem to show that $\mathbb{E}[\frac{1}{T+2}] = \frac{1}{4}$.

4. Let S and T be stopping times with respect to the filtration $\mathcal{F}_n$.
   (a) Show that $\min(S, T)$ and $\max(S, T)$ are stopping times.
   (b) Suppose $S \le T$. Is it necessarily true that $T - S$ is a stopping time?

5. Suppose that we repeatedly toss a fair coin, writing H for heads and T for tails. What is the expected number of tosses until we have seen the pattern $HTHT$ for the first time?
Give an example of a four letter pattern of $\{H, T\}$ that has the maximal expected number of tosses, of any four letter pattern, until it is seen.

6. Let $m \in \mathbb{N}$ and $m \ge 2$. At time $n = 0$, an urn contains $2m$ balls, of which m are red and m are blue. At each time $n = 1, \dots, 2m$ we draw a single ball from the urn; we do not replace it. Therefore, at time n the urn contains $2m - n$ balls.
Let N_n denote the number of red balls remaining in the urn at time n. For $n = 0, \dots, 2m - 1$ let
$$P_n = \frac{N_n}{2m - n}$$
be the fraction of red balls remaining after time n. Let $\mathcal{G}_n = \sigma(N_i\,;\, i \le n)$.
   (a) Show that P_n is a $\mathcal{G}_n$ martingale.
   (b) Let T be the first time at which the ball that we draw is red. Note that $T < 2m$, because the urn initially contains at least 2 red balls. Show that the probability that the $(T + 1)^{st}$ ball is red is $\frac{1}{2}$.

Problem Sheet 3

1. You play a game by betting on outcome of i.i.d. random variables X_n, $n \in \mathbb{Z}^+$, where

$$\mathbb{P}[X_n = 1] = p, \quad \mathbb{P}[X_n = -1] = q = 1 - p, \quad \frac{1}{2} < p < 1.$$

Let Z_n be your fortune at time n, that is $Z_n = Z_0 + \sum_{j=1}^{n} C_j X_j$. The bet C_n you place on game n must be in $(0, Z_{n-1})$ (i.e. you cannot borrow money to place bets). Your objective is to maximise the expected 'interest rate' $\mathbb{E}[\log(Z_N/Z_0)]$, where N (the length of the game) and Z_0 (your initial fortune) are both fixed. Let $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$. Show that if C is a previsible strategy, then $\log Z_n - n\alpha$ is a supermartingale, where

$$\alpha = p \log p + q \log q + \log 2,$$

and deduce that $\mathbb{E} \log(Z_n/Z_0) \le N\alpha$.

Can you find a strategy such that $\log Z_n - n\alpha$ is a martingale?

2. Let $\mathcal{F}_n$ be a filtration. Suppose T is a stopping time such that for some $K \ge 1$ and $\epsilon > 0$ we have, for all $n \ge 0$, almost surely

$$\mathbb{P}\left[T \le n + K \,|\, \mathcal{F}_n\right] \ge \epsilon.$$

   (a) Prove by induction that for all $m \in \mathbb{N}$, $\mathbb{P}\left[T \ge mK\right] \le (1 - \epsilon)^m$.

   (b) Hence show that $\mathbb{E}[T] < \infty$.

3. Let $X_1, X_2, \ldots$ be a sequence of iid random variables with

$$\mathbb{P}[X_1 = 1] = p, \quad \mathbb{P}[X_1 = -1] = q, \quad \text{where } 0 < p = 1 - q < 1,$$

and suppose that $p \ne q$. Let $a, b \in \mathbb{N}$ with $0 < a < b$, and let

$$S_n = a + X_1 + \ldots + X_n, \quad T = \inf\{n \ge 0\,;\, S_n = 0 \text{ or } S_n = b\}.$$

Let $\mathcal{F}_n = \sigma(X_i\,;\, i \le n)$.

   (a) Deduce from the previous question that $\mathbb{E}[T] < \infty$.

   (b) Show that

$$M_n = \left(\frac{q}{p}\right)^{S_n}, \qquad N_n = S_n - n(p - q)$$

   are both $\mathcal{F}_n$ martingales.

   (c) Calculate $\mathbb{E}[S_T]$, $\mathbb{P}[S_T = 0]$ and hence calculate $\mathbb{E}[T]$.

4. Let $X_1, X_2, \ldots$ be strictly positive iid random variables such that $\mathbb{E}[X_1] = 1$ and $\mathbb{P}[X_1 = 1] < 1$.

   (a) Show that $M_n = \prod_{i=1}^{n} X_i$ is a martingale relative to a natural filtration that you should specify.

   (b) Deduce that there exists a real valued random variable L such that $M_n \to L$ almost surely as $n \to \infty$.

   (c) Show that $\mathbb{P}[L = 0] = 1$.
   Hint: Argue by contradiction and note that if $M_n, M_{n+1} \in (c-\epsilon, c+\epsilon)$ *then* $X_{n+1} \in (\frac{c-\epsilon}{c+\epsilon}, \frac{c+\epsilon}{c-\epsilon})$.

   (d) Use the Strong Law of Large Numbers to show that there exists $c \in \mathbb{R}$ such that $\frac{1}{n} \log M_n \to c$ almost surely $n \to \infty$. Use Jensen's inequality to show that $c < 0$.

5. Show that a set $\mathcal{C}$ of random variables is uniformly integrable if either:

   (a) There exists a random variable Y such that $\mathbb{E}[|Y|] < \infty$ and $|X| \le Y$ for all $X \in \mathcal{C}$.

   (b) There exists $p > 1$ and $A < \infty$ such that $\mathbb{E}[|X|^p] \le A$ for all $X \in \mathcal{C}$.

6. Let Z_n be a Galton-Watson process with offspring distribution G (which takes value in $0, 1, \ldots$), where $\mathbb{E}[G] = \mu > 1$ and $\mathrm{var}[G] = \sigma^2 < \infty$. Set $M_n = \frac{Z_n}{\mu^n}$, and use the filtration from lecture notes.

Show that M_n is a martingale. Using induction, find a formula $\mathbb{E}[M_n^2]$ in terms of n, μ and σ. Hence, show that $\sup_{n \in \mathbb{N}} \mathbb{E}[M_n^2] < \infty$ and that M_n converges almost surely and in L^1 as $n \to \infty$. Deduce that the limit M_∞ satisfies $\mathbb{P}[M_\infty > 0] > 0$.